

An asymptotically distribution-free test of symmetry

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Abstract

A procedure, based on sample spacings, is proposed for testing whether a univariate distribution is symmetric about some unknown value. The proposed test is a modification of a sign test suggested by Antille and Kersting [1977. Tests for symmetry. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 39, 235–255], but unlike Antille and Kersting's test, our modified test is asymptotically distribution-free and is usable in practice. A simulation study indicates that the proposed test maintains the nominal level of significance, α fairly accurately even for samples of size as small as 20, and a comparison with the classical test based on sample coefficient of skewness, shows that our test has good power for detecting different asymmetric distributions.

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1. Introduction

Many statistical procedures assume that data come from a normal distribution. However, many such procedures are robust to violations of normality, so that having data from a symmetric distribution is often sufficient for their validity. Other procedures, such as nonparametric methods, assume symmetric distributions rather than normal distributions. The presence or absence of symmetry is also important in terms of deciding what parameter to estimate. Bickel and Lehmann (1975) and Antille et al. (1982), among others, argue that if F is symmetric, the point of symmetry μ is the only natural measure of location, whereas if F is nonsymmetric there is no longer only one reasonable measure of location.

Thus, there are many reasons for investigating the presence or absence of symmetry, and the problem of testing symmetry has been receiving much attention in the literature; see, e.g., Gupta (1967), Gastwirth (1971), Antille and Kersting (1977), Feuerverger and Mureika (1977), Randles et al. (1980), Antille et al. (1982), Csörgő and Heathcote (1987), Cabilio and Masaro (1996), and Mira (1999). The goal is to design tests that (a) have good power for interesting alternatives to symmetry and (b) are either asymptotically distribution-free or at least have a type-I error probability that stays relatively constant for distributions satisfying the null hypothesis of symmetry.

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In the current article we suggest a modification of a sign test based on sample spacings suggested by Antille and Kersting (1977). The asymptotic variance of Antille and Kersting's sign statistic depends on the underlying distribution function, so that the test is valid for a given family of distributions and is not asymptotically distribution-free. Further, the asymptotic variance is almost three times larger for the Cauchy distribution than for the normal, so the type-I error probability does not stay relatively constant. In order to obtain a limiting distribution that does not depend on the underlying distribution we suggest a consistent estimator of the asymptotic variance. Our modified test is defined in the next section, where we provide results for the asymptotic distribution of the test statistic under the null hypothesis of symmetry, as well as under a sequence of converging alternatives of the form $F(t + n^{-1/2}\gamma(t))$, where F is symmetric about zero, γ is some smooth function, and n is the sample size. The estimator of the asymptotic variance is shown to be consistent in Section 3, and Section 4 contains a simulation study where our modified test is compared with the classical test based on the skewness coefficient (Gupta, 1967). The proofs are given in the Appendix.

2. A test for symmetry about an unknown value

Assume that X_1, \dots, X_n are independent random variables (rvs) with distribution function F and density function f . Denote the order statistics by $X_{(1)}, \dots, X_{(n)}$ and the "first-order spacings" by

$$D_i = X_{(i+1)} - X_{(i)}, \quad i = 1, \dots, n-1.$$

In Antille and Kersting (1977) (hereafter referred to as AK in this article), a sign test statistic

$$S = n^{-1/2} \sum_{i=1}^{[(n-1)/2]} \left(I_{\{D_i - D_{n-i} \leq 0\}} - \frac{1}{2} \right)$$

based on the spacings $\{D_i\}$, is proposed for tests for symmetry of F about an unknown value μ , where $[x]$ denotes the greatest integer smaller than or equal to x . Due to technical problems arising from the behavior of the derivatives of $\Psi = F^{-1}$ near 0 and 1, AK derived asymptotic properties for the statistic

$$S_\varepsilon = n^{-1/2} \sum_{i=1+[\varepsilon n]}^{[(n-1)/2]} \left(I_{\{D_i - D_{n-i} \leq 0\}} - \frac{1}{2} \right)$$

rather than for S ; the statistic S_ε can be regarded as a trimmed version of S . A problem is that although S_ε is asymptotically normal, it does not provide an asymptotically distribution-free test for symmetry. This is because its asymptotic variance depends on the underlying distribution (in a rather complicated way). This problem, however, can be circumvented by finding a nonparametric and consistent estimator of the variance.

The asymptotic variance of S_ε is given by (see AK)

$$\sigma_\varepsilon^2 = \frac{1}{16} + \frac{1}{16} \int_\varepsilon^{1-\varepsilon} \left(1 - \log \frac{\Psi'(t)}{\Psi'(1/2)} \right)^2 dt + \frac{\varepsilon}{16} \left(\log \frac{\Psi'(\varepsilon)}{\Psi'(1/2)} \right)^2 - \frac{\varepsilon}{8}.$$

Further, the limit $\sigma^2 = \lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon^2$ is equal to

$$\frac{1}{16} + \frac{1}{16} \int_0^1 \left(1 - \log \frac{\Psi'(t)}{\Psi'(1/2)} \right)^2 dt = \frac{1}{16} + \frac{1}{16} \int_{-\infty}^{\infty} \left(1 + \log \frac{f(t)}{f(\mu)} \right)^2 dF(t), \quad (1)$$

provided that the integrals exist.

Motivated by the ideas of Vasicek (1976) and van Es (1992), we construct an estimator of σ^2 by replacing the distribution function on the left-hand side of (1) by the empirical distribution function, and using a difference operator in place of the differential operator. The derivative of the inverse of F , $\Psi'(t)$, is then estimated by $nD_{i,m}/(2m)$ for $i/n \leq t < (i+1)/n$, $i = m, \dots, n-m$, where $D_{i,m} = X_{(i+m)} - X_{(i-m+1)}$ are the spacings of order $2m-1$.

If we assume that $m/n \rightarrow 0$ as $m, n \rightarrow \infty$, then this motivates the following estimator of σ^2 :

$$\hat{\sigma}^2 = \frac{1}{16} + \frac{1}{16(n - 2m + 1)} \sum_{i=m}^{n-m} \left(1 - \log \frac{D_{i,m}}{D_{k,m}} \right)^2,$$

where $k = [n/2]$. Likewise, we estimate σ_ε^2 by

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{16} + \frac{1}{16(n - 2(m + [\varepsilon n]) + 1)} \sum_{i=m+[\varepsilon n]}^{n-m-[\varepsilon n]} \left(1 - \log \frac{D_{i,m}}{D_{k,m}} \right)^2 + \frac{\varepsilon}{16} \left(\log \frac{D_{[\varepsilon n],m}}{D_{k,m}} \right)^2 - \frac{\varepsilon}{8}.$$

In order to obtain *asymptotically distribution-free* tests for symmetry, we suggest the test statistics

$$T = S/\hat{\sigma} \quad \text{and} \quad T_\varepsilon = S_\varepsilon/\hat{\sigma}_\varepsilon.$$

From Theorem 3 in AK, together with our Theorem 3 stated in the next section (on the consistency of the above variance estimators), we obtain the following basic result.

Theorem 1. *Assume that X_1, \dots, X_n are independent rvs from F , where F is symmetric. Further, assume that the support of F is a (possibly infinite) interval and that f is strictly positive and twice continuously differentiable on this interval. Then T_ε is asymptotically $N(0, 1)$.*

Thus, to test the null hypothesis of symmetry,

$$H_0 : F(t - \mu) = 1 - F(-(t - \mu)) \quad \text{for all } t,$$

versus the general alternative of asymmetry, corresponding to

$$H_1 : F(t - \mu) \neq 1 - F(-(t - \mu)) \quad \text{for at least one } t,$$

at the approximate (n large) α level of significance, we reject H_0 whenever $|T_\varepsilon| > z_{\alpha/2}$, where z_α denotes the α quantile of the standard normal distribution. We conjecture that this test procedure is asymptotically valid not only for T_ε , but for T as well. This is confirmed by the simulations in Section 4, although we do not have a formal proof of the latter assertion as yet.

The asymptotic distribution of T_ε under close alternatives is studied next. As in AK, we consider close alternatives of the form $G_n(t) = F(t + n^{-1/2}\gamma(t))$ and assume, without loss of generality, that F is symmetric about 0. The function $t + n^{-1/2}\gamma(t)$ is assumed to be a monotonously increasing function of t for large n . By Theorem 5 in AK and Theorem 3 in the next section, we obtain the following theorem on the asymptotic distribution of T_ε under the close alternatives considered.

Theorem 2. *Assume that X_1, \dots, X_n are independent rvs from G_n and that γ is twice continuously differentiable with bounded derivatives. Then, under the assumptions on F made in Theorem 1, T_ε is asymptotically $N(v_\varepsilon\sigma_\varepsilon^{-1}, 1)$, where*

$$v_\varepsilon = \frac{1}{4} \int_\varepsilon^{1/2} (\gamma'(\Psi(t)) - \gamma'(-\Psi(t))) dt.$$

The limit $v = \lim_{\varepsilon \rightarrow 0} v_\varepsilon$ exists and is equal to

$$v = \frac{1}{4} \int_{-\infty}^0 (\gamma'(t) - \gamma'(-t)) dF(t).$$

Remark (asymptotic relative efficiency). The Pitman asymptotic relative efficiency (ARE) of a test relative to another, is defined as the limit of the inverse ratio of sample sizes required to obtain the same limiting power, at a sequence of alternatives converging to the null hypothesis. If the limiting power of a test at a sequence of alternatives converges to a number in the open interval from the level, α , to the maximum power viz. 1, then a measure of rate of convergence,

called “efficacy” can be computed. Under some standard regularity conditions (see e.g. Fraser, 1957), which include the asymptotic normality of the test under the null and under the sequence of alternatives, this is given by

$$\text{efficacy} = \frac{\mu^2}{\sigma^2},$$

where μ and σ^2 are the mean and variance under the alternative sequence, when the statistic has been normalized to have a limiting standard normal distribution under the null.

Thus, from Theorems 1 and 2, one can see that the “efficacy” of the test statistic T_ε is $(v_\varepsilon/\sigma_\varepsilon)^2$, which coincides with that for the test statistic S_ε . In AK, values of $\lim_{\varepsilon \rightarrow 0} (|v_\varepsilon|/\sigma_\varepsilon)$ are given for the normal, double exponential, logistic, and Cauchy distributions. AK also computed values of (the square root of the) efficacy for some other tests, and concluded that the test based on S_ε is a serious competitor among scale and translation invariant tests for symmetry. These asymptotic conclusions still hold for our modified distribution-free test.

3. Consistency of variance estimators

Assume that X_1, \dots, X_n are independent rvs with distribution function $G_n(t) = F(t + n^{-1/2}\gamma(t))$ (note: we do not require F to be symmetric in this section). Again, due to technical problems arising from the behavior of the derivatives of $\Psi = F^{-1}$ near 0 and 1, we will show consistency for

$$\tilde{\sigma}_\delta^2 = \frac{1}{16} + \frac{1}{16(n - 2(m + [\delta n]) + 1)} \sum_{i=m+[\delta n]}^{n-m-[\delta n]} \left(1 - \log \frac{D_{i,m}}{D_{k,m}}\right)^2$$

rather than for $\hat{\sigma}^2$, where δ is a small positive number. In the next theorem we show that $\hat{\sigma}_\varepsilon^2$ and $\tilde{\sigma}_\delta^2$ are consistent estimators of σ_ε^2 and

$$\sigma_\delta^2 = \frac{1}{16} + \frac{1}{16} \int_{\Psi(\delta)}^{\Psi(1-\delta)} \left(1 + \log \frac{f(t)}{f(\mu)}\right)^2 dF(t),$$

respectively. Note that σ_δ^2 can be made arbitrarily close to σ^2 by choosing δ sufficiently small (provided that the integrals (1) exist).

Theorem 3. Assume that $m/n \rightarrow 0$ and $m/\log n \rightarrow \infty$ as $n \rightarrow \infty$. Then, under the assumptions of Theorem 2 (except the assumption on symmetry of F), $\hat{\sigma}_\varepsilon^2 \rightarrow \sigma_\varepsilon^2$ and $\tilde{\sigma}_\delta^2 \rightarrow \sigma_\delta^2$ a.s. as $n \rightarrow \infty$.

The proof of Theorem 3 which is given in the Appendix, relies on the following general result, which is also proved in the Appendix.

Theorem 4. Assume that h is a continuous function on $(0, \infty)$. Then, under the assumptions of Theorem 3,

$$\lim_{n \rightarrow \infty} \frac{1}{n - 2(m + [\delta n]) + 1} \sum_{i=m+[\delta n]}^{n-m-[\delta n]} h\left(\frac{2m}{nD_{i,m}}\right) = \int_{\Psi(\delta)}^{\Psi(1-\delta)} h(f(t)) dF(t) \quad \text{a.s.}$$

4. Simulation study

In this section we consider T rather than T_ε since we believe that T is the appropriate test statistic. Three basic questions will be addressed:

- (1) How to choose the value of m which provides a good estimator of $\hat{\sigma}^2 = \hat{\sigma}^2(m)$?
- (2) How do the level (actual versus nominal) and power of the test $T = S/\hat{\sigma}$ behave for different underlying models F ?
- (3) How powerful is T for detecting asymmetry?

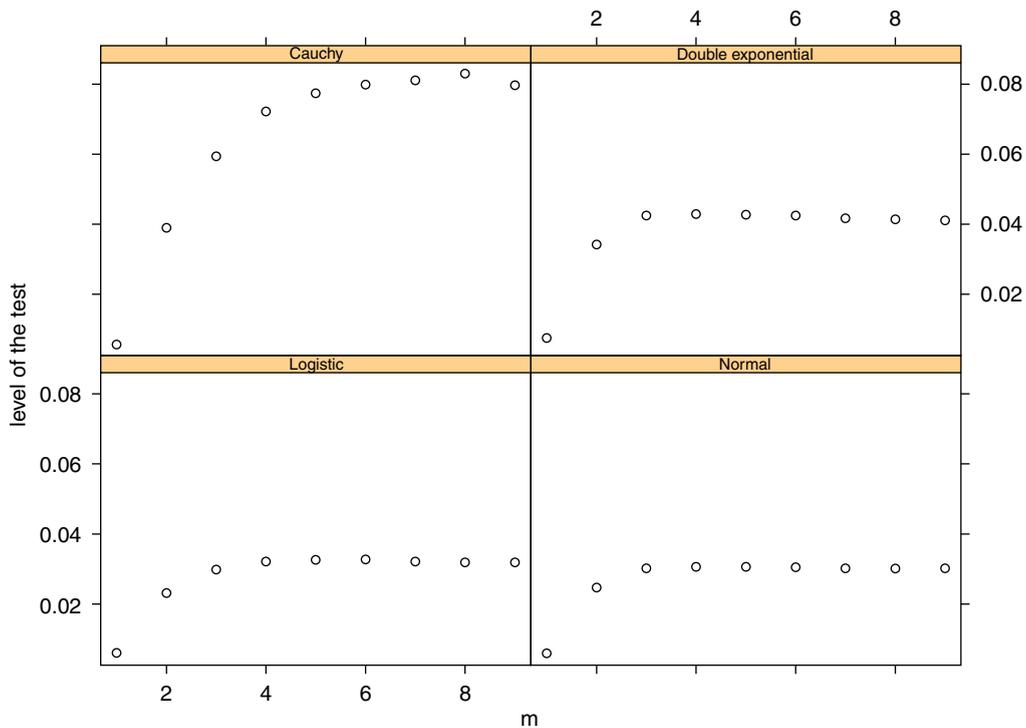


Fig. 1. The level of the test as a function of m . Here $n = 20$ and $\alpha = 0.05$.

A competitor to the test T is the classical test of skewness (Gupta, 1967), which is based on the sample coefficient of skewness

$$b_1^{1/2} = M_3 M_2^{-3/2},$$

where M_k is the k th sample moment about the sample mean. This statistic tests the null hypothesis that the coefficient of skewness $\mu_3 \mu_2^{-3/2}$, with μ_k denoting the k th central moment of the underlying distribution, is equal to zero versus the alternative that it is not equal to zero. Under the null hypothesis, and if $\mu_6 < \infty$, $(nb_1/\tau^2)^{1/2}$ has the limit distribution $N(0, 1)$, where $\tau^2 = (\mu_6 - 6\mu_2\mu_4 + 9\mu_2^3)/\mu_2^3$. An asymptotically distribution-free test is obtained by replacing the central moments in τ^2 by the corresponding sample central moments. See Gupta (1967) for details.

In the simulation study we focus on four different distributions F :

- Cauchy, $f(t) = \pi^{-1}(1 + t^2)^{-1}$;
- double exponential, $f(t) = 2^{-1}e^{-|t|}$;
- logistic, $f(t) = e^{-t}(1 + e^{-t})^{-2}$; and
- normal, $f(t) = (2\pi)^{-1/2}e^{-t^2/2}$.

The alternative considered is $G_n(t) = F(t + n^{-1/2}\gamma_a(t))$, where a is a nonnegative constant and $\gamma_a(t) = at, t \geq 0$, and 0 elsewhere. Each entry in the tables and the figures is based on 10,000 replications.

In Figs. 1–4 we see how the actual level of the test T depends on the underlying distribution and the choice of m , when the nominal level of test is $\alpha = 0.05$. From the figures we conclude that m should be kept relatively small and that m should increase slowly with the sample size n in order to get the actual level of the test close to the nominal level. For n in the range 20–100, $m = 3$ appears to be a reasonable choice, while for $n = 500$, the value of m needs to be a bit larger, e.g., equal to 4 or 5. In Table 1, estimates of the actual levels are given for the different distributions

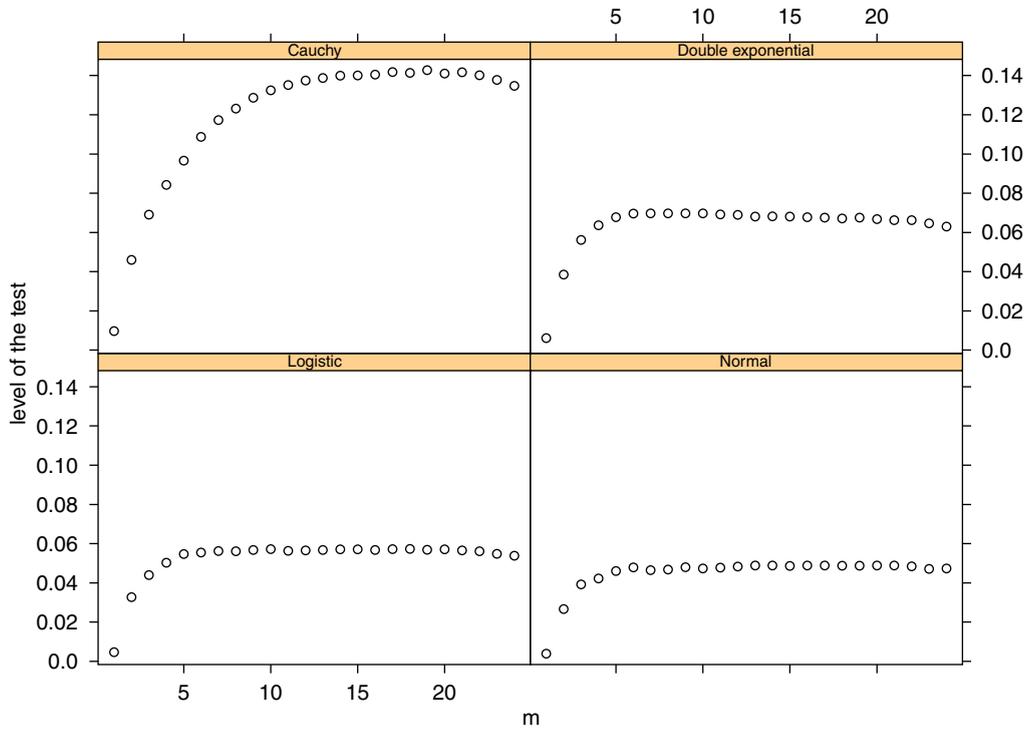


Fig. 2. The level of the test as a function of m . Here $n = 50$ and $\alpha = 0.05$.

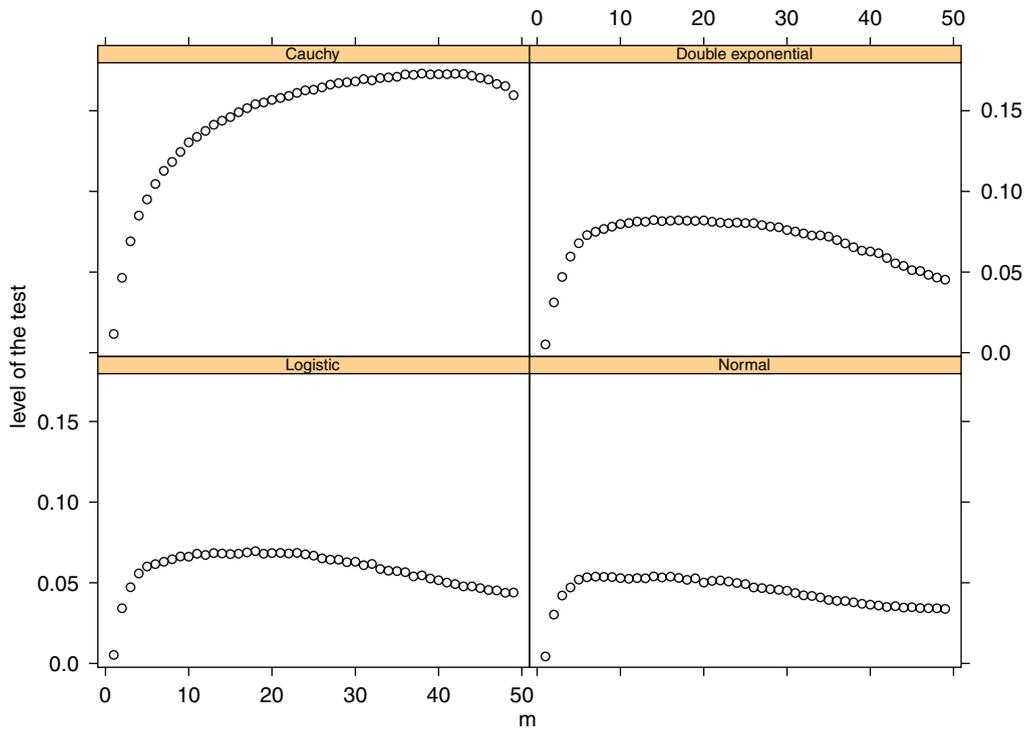


Fig. 3. The level of the test as a function of m . Here $n = 100$ and $\alpha = 0.05$.

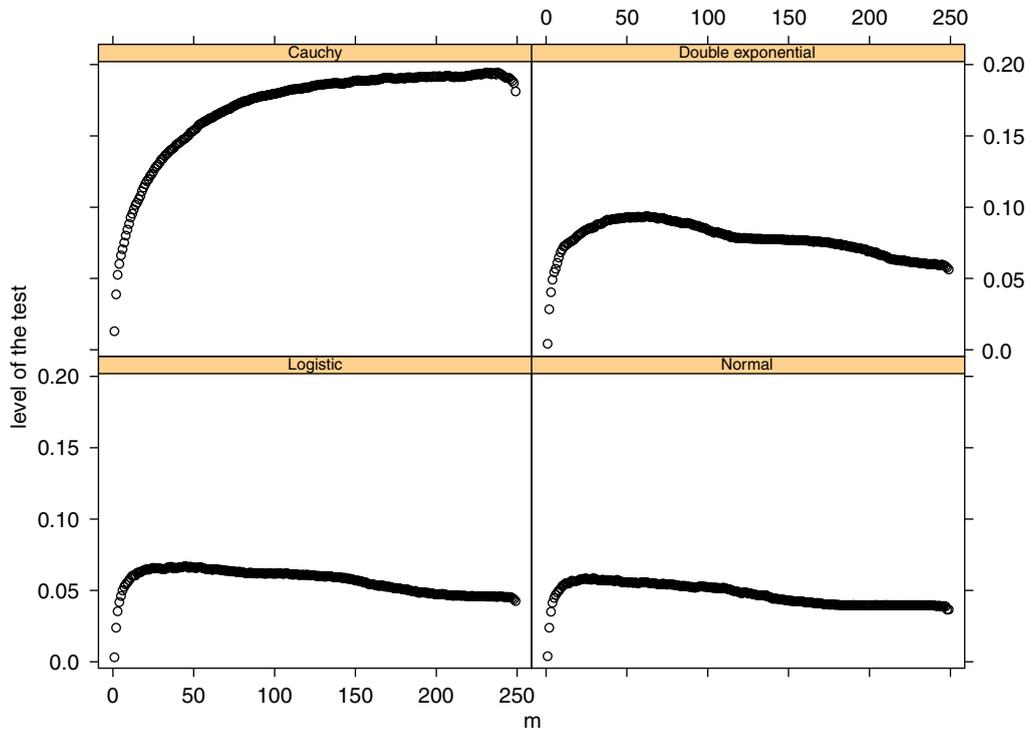


Fig. 4. The level of the test as a function of m . Here $n = 500$ and $\alpha = 0.05$.

Table 1
Estimates of the actual level of the test T

n	m	Distribution			
		Cauchy	Double exponential	Logistic	Normal
20	3	0.059	0.043	0.030	0.030
50	3	0.069	0.056	0.044	0.039
100	3	0.069	0.047	0.047	0.042
500	4	0.060	0.049	0.042	0.041
500	5	0.066	0.054	0.046	0.044

Table 2
Estimates of the actual level of the classical test of skewness

n	Distribution		
	Double exponential	Logistic	Normal
20	0.047	0.028	0.026
50	0.046	0.035	0.035
100	0.043	0.041	0.042
500	0.042	0.042	0.046

and some different sample sizes, with corresponding suggested values of m . We see that the actual levels stay relatively constant, although it is noticeable that the level is lower for $n = 20$ and higher for the Cauchy distribution than for the other cases. Further, the actual levels of the test T in Table 1 are close to the corresponding levels of the classical test of skewness (Table 2).

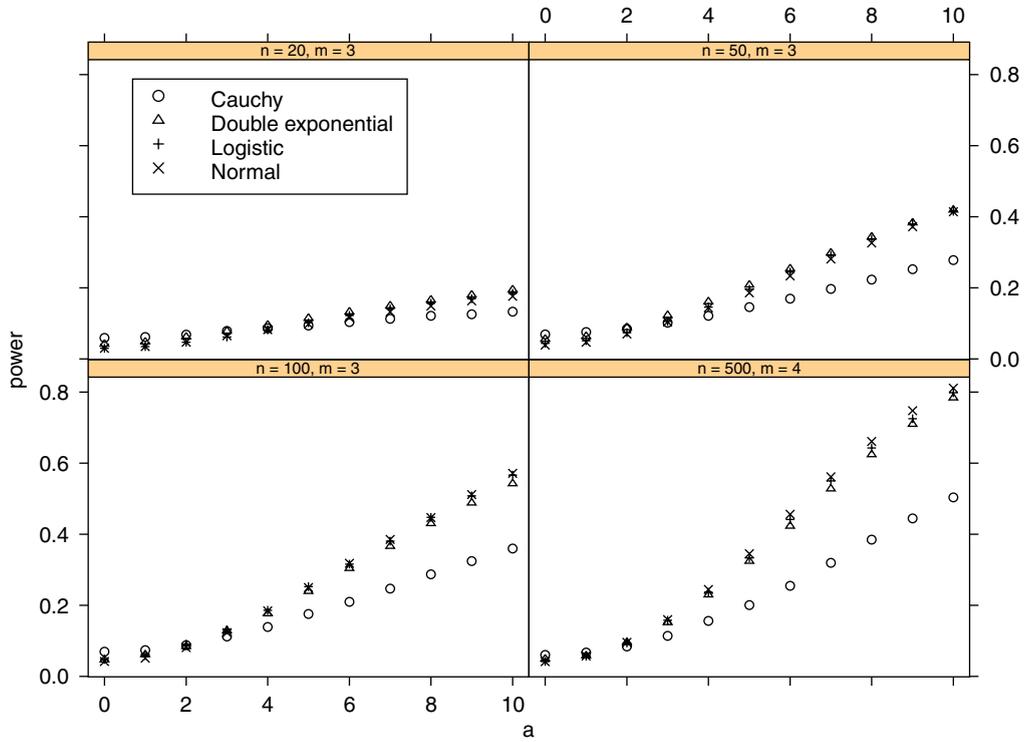


Fig. 5. The power of the test T as a function of a . Here $\alpha = 0.05$.

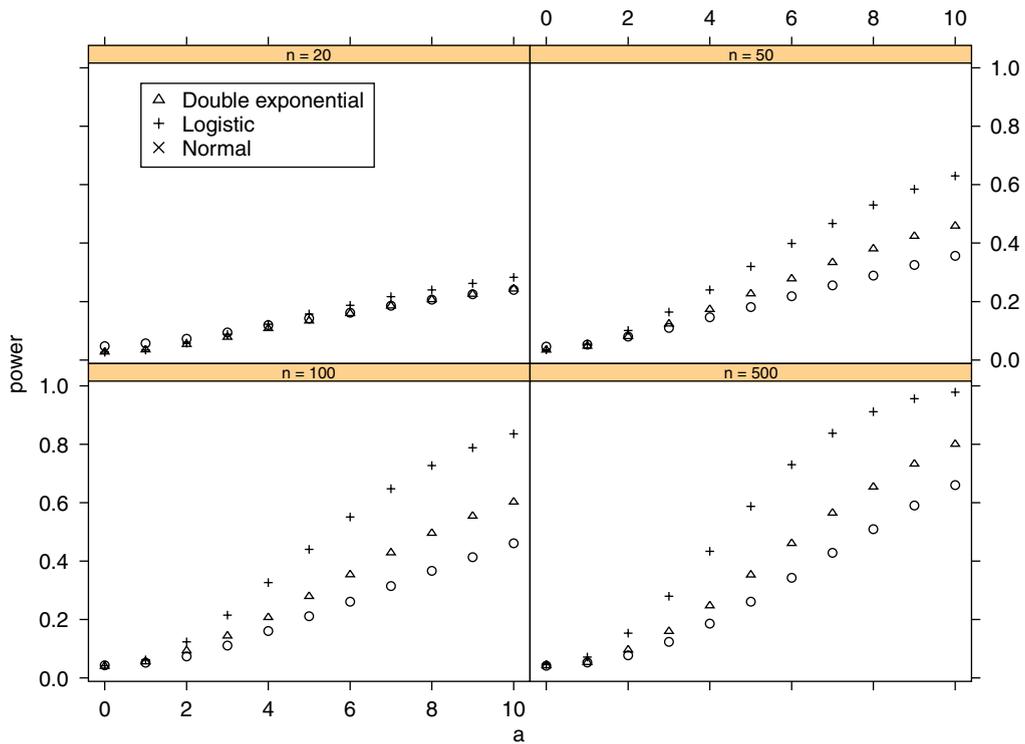


Fig. 6. The power of the classical test of skewness as a function of a . Here $\alpha = 0.05$.

Next, consider Figs. 5 and 6. In Fig. 5, which gives the power of the test T as a function of a (where a determines the alternative), we see that the power against close alternatives is very close for the normal, the logistic and the double exponential distribution. For the Cauchy distribution the power is somewhat lower than for the other three distributions. For the classical test the power depends heavily on the underlying distribution. When F is normal the classical test has more power than the test T (which is of no surprise since tests based on $b_1^{1/2}$ are often referred to as ‘tests of normality’). If F is the logistic distribution then the results of the two tests are quite close, while if F is double exponential, then the test T is the winner (except when $n = 20$).

Remark. Recall that the classical test is not valid for heavy-tailed distributions such as the Cauchy for which the moments do not exist, whereas our test is. Secondly, it should be noted that there exist asymmetrical distributions for which the third central moment μ_3 is zero (see e.g. MacGillivray, 1982), and that the classical test is unable to detect such alternatives, whereas our test can.

In view of the simulations as well as the above Remark, one may conclude that the test T is a serious competitor to the classical test, and is a valid test of symmetry in a wider range of situations including for those distributions with heavy tails.

Appendix A

Lemma 1 (Glivenko (1933), Cantelli (1933)). *If H_n is the empirical distribution function of a sample U_1, \dots, U_n of uniform rvs on $(0, 1)$, then*

$$\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} |i/n - U_{(i)}| \leq \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |H_n(t) - t| = 0 \quad a.s. \tag{A.1}$$

Remark. The inequality in (A.1) follows from the fact that $H_n(U_{(i)}) = i/n$.

Lemma 2 (Yu and Ekström, 2000). *Let U_1, \dots, U_n be uniform rvs on $(0, 1)$ and assume that $m/n \rightarrow 0$ and $m/\log n \rightarrow \infty$ as $n \rightarrow \infty$. Then with probability 1, for any fixed $0 < \varepsilon < 1$ and n sufficiently large,*

$$\max_{1 \leq i \leq n-m} (U_{(i+m)} - U_{(i)}) \leq (1 + \varepsilon)m/n.$$

Lemma 3 (Yu, 1986, 1994). *Assume that X_1, \dots, X_n are independent rvs with density function f . If $m/n \rightarrow 0$ and $m/\log n \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$f_n(t) = \begin{cases} 2m/(nD_{i,m}) & \text{if } t \in [X_{(i)}, X_{(i+1)}), \quad i = m, \dots, n - m, \\ 0 & \text{otherwise,} \end{cases}$$

tends a.s. to $f(t)$ for a.a. $t \in \mathbb{R}$ as $n \rightarrow \infty$. If, in addition, f is positive and continuous on a compact set K , then

$$\lim_{n \rightarrow \infty} \sup_{t \in K} |f_n(t) - f(t)| = 0 \quad a.s.$$

Lemma 4. *Lemma 3 remains valid if X_1, \dots, X_n are independent rvs with distribution function $G_n(t) = F(t + n^{-1/2}\gamma(t))$, where the density function f corresponding to F exists and where γ is twice continuously differentiable with bounded derivatives (F is not required to be symmetric).*

Proof of Theorem 4. Let $U_i = G_n(X_i)$, $i = 1, \dots, n$, and note that U_1, \dots, U_n are independent and uniformly distributed on $(0, 1)$. Denote the inverse of G_n by Ψ_n . By AK (p. 245),

$$\Psi_n(t) = \Psi(t) - n^{-1/2}\gamma(\Psi(t)) - n^{-1}\beta_n(\Psi(t)), \tag{A.2}$$

where β_n and γ are, for n large enough, uniformly bounded on compact subsets of R . Let ρ be a small positive number, $0 < \rho < \delta/2$, and define $m_\delta = m + [\delta n]$. By Lemma 1,

$$P\left(\max_{1 \leq i \leq n} |U_{(i)} - i/n| > \rho \text{ i.o.}\right) = 0.$$

Thus, there exists (a random) N such that for all $n \geq N$,

$$\frac{m_\delta}{n} - \rho \leq U_{(m_\delta)} \leq U_{(n-m_\delta)} \leq \frac{n - m_\delta}{n} + \rho$$

holds with probability 1, which together with (A.2) imply that

$$-\infty < \Psi(\delta - 2\rho) \leq \Psi_n\left(\frac{m_\delta}{n}\right) \leq X_{(m_\delta)} \leq X_{(n-m_\delta)} \leq \Psi_n\left(\frac{n - m_\delta}{n} + \rho\right) \leq \Psi(1 - \delta + 2\rho) < \infty$$

holds with probability 1 for n sufficiently large. Let $K = [\Psi(\delta - 2\rho), \Psi(1 - \delta + 2\rho)]$. Then, by Lemma 4,

$$\lim_{n \rightarrow \infty} \max_{m_\delta \leq i \leq n-m_\delta} |f_n(X_{(i)}) - f(X_{(i)})| \leq \lim_{n \rightarrow \infty} \sup_{t \in K} |f_n(t) - f(t)| = 0 \quad \text{a.s.}$$

and moreover, since h is uniformly continuous on K ,

$$\lim_{n \rightarrow \infty} \max_{m_\delta \leq i \leq n-m_\delta} |h(f_n(X_{(i)})) - h(f(X_{(i)}))| \leq \lim_{n \rightarrow \infty} \sup_{t \in K} |h(f_n(t)) - h(f(t))| = 0 \quad \text{a.s.} \tag{A.3}$$

Further, by Lemma 1 and (A.2),

$$\begin{aligned} &\lim_{n \rightarrow \infty} \max_{m_\delta \leq i \leq n-m_\delta} |h(f(X_{(i)})) - h(f(\Psi(i/n)))| \\ &\leq \lim_{n \rightarrow \infty} \max_{m_\delta \leq i \leq n-m_\delta} |h(f(\Psi_n(U_{(i)}))) - h(f(\Psi_n(i/n)))| \\ &\quad + \lim_{n \rightarrow \infty} \max_{m_\delta \leq i \leq n-m_\delta} |h(f(\Psi(i/n))) - h(f(\Psi_n(i/n)))| = 0, \quad \text{a.s.} \end{aligned} \tag{A.4}$$

and by (A.3) and (A.4) and the triangle inequality,

$$\lim_{n \rightarrow \infty} \max_{m_\delta \leq i \leq n-m_\delta} |h(f_n(X_{(i)})) - h(f(\Psi(i/n)))| = 0 \quad \text{a.s.}$$

Thus,

$$\begin{aligned} \frac{1}{n - 2m_\delta + 1} \sum_{i=m_\delta}^{n-m_\delta} h(f_n(X_{(i)})) &= \frac{1}{n - 2m_\delta + 1} \sum_{i=m_\delta}^{n-m_\delta} h(f(\Psi(i/n))) \\ &\quad + \frac{1}{n - 2m_\delta + 1} \sum_{i=m_\delta}^{n-m_\delta} (h(f_n(X_{(i)})) - h(f(\Psi(i/n)))) \\ &\rightarrow \int_{\delta}^{1-\delta} h(f(\Psi(s))) \, ds = \int_{\Psi(\delta)}^{\Psi(1-\delta)} h(f(t)) \, dF(t) \quad \text{a.s.} \end{aligned}$$

as $n \rightarrow \infty$, as was to be proved. \square

Proof of Theorem 3. Let $U_i = G_n(X_i)$, $i = 1, \dots, n$, and note that U_1, \dots, U_n are independent and uniformly distributed on $(0, 1)$. Denote the inverse of G_n by Ψ_n . By AK (p. 245),

$$\Psi_n(t) = \Psi(t) - n^{-1/2}\gamma(\Psi(t)) - n^{-1}\beta_n(\Psi(t)),$$

where β_n is, for n large enough, uniformly bounded on compact subsets of R . Thus, by Lemmata 1 and 4,

$$2m/(nD_{k,m}) = f_n(X_{(k)}) = [f_n(X_{(k)}) - f(X_{(k)})] + [f(\Psi_n(U_{(k)})) - f(\Psi_n(k/n))] + f(\Psi_n(k/n)) \rightarrow f(\Psi(1/2)) = f(\mu) \quad \text{a.s.}$$

as $n \rightarrow \infty$. This result together with Theorem 4 imply the desired result. \square

Proof of Lemma 4. Let $Y_i = \Psi(G_n(X_i))$ and $U_i = F(Y_i)$, $i = 1, \dots, n$, and note that Y_1, \dots, Y_n are independent rvs from F and that U_1, \dots, U_n are independent uniform rvs on $(0, 1)$.

Define $y = y(x)$ (depending on n) by

$$y = x + n^{-1/2}\gamma(x) = x + n^{-1/2}\gamma(y) + n^{-1}\beta_n(y),$$

where $\beta_n(y) = n^{1/2}(\gamma(x) - \gamma(y))$. As is shown in AK, β'_n is, for n large enough, uniformly bounded on compact subsets of R . Further, by the definition of Y_i ,

$$X_i = Y_i - n^{-1/2}\gamma(Y_i) - n^{-1}\beta_n(Y_i).$$

Thus,

$$D_{i,m} = X_{(i+m)} - X_{(i-m+1)} = Y_{(i+m)} - Y_{(i-m+1)} - n^{-1/2}(\gamma(Y_{(i+m)}) - \gamma(Y_{(i-m+1)})) - n^{-1}(\beta_n(Y_{(i+m)}) - \beta_n(Y_{(i-m+1)})),$$

and

$$\frac{2m}{nD_{i,m}} = \frac{2m}{n\tilde{D}_{i,m}} \left/ \left[1 - \frac{1}{n^{1/2}}\gamma_i^* - \frac{1}{n}\beta_{n,i}^* \right] \right., \tag{A.5}$$

where $\tilde{D}_{i,m} = Y_{(i+m)} - Y_{(i-m+1)}$,

$$\gamma_i^* = \frac{\gamma(Y_{(i+m)}) - \gamma(Y_{(i-m+1)})}{Y_{(i+m)} - Y_{(i-m+1)}} \quad \text{and} \quad \beta_{n,i}^* = \frac{\beta_n(Y_{(i+m)}) - \beta_n(Y_{(i-m+1)})}{Y_{(i+m)} - Y_{(i-m+1)}}.$$

By assumption, Ψ' is uniformly bounded on compact subsets of R . Thus, Lemma 2 and the mean value theorem imply that with probability 1, for some constant $c > 0$ and n sufficiently large,

$$\tilde{D}_{i,m} = \Psi(U_{(i+m)}) - \Psi(U_{(i-m+1)}) = \Psi'(\tilde{U}_i)(U_{(i+m)} - U_{(i-m+1)}) \leq cm/n,$$

where $U_{(i-m+1)} \leq \tilde{U}_i \leq U_{(i+m)}$. Therefore, since both γ' and β'_n are uniformly bounded on compact subsets of R , we see that with probability 1, both γ_i^* and $\beta_{n,i}^*$ are uniformly bounded for n sufficiently large. This together with equality (A.5) and Lemma 3 complete the proof. \square

References

Antille, A., Kersting, G., 1977. Tests for symmetry. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 39, 235–255.
 Antille, A., Kersting, G., Zucchini, W., 1982. Testing symmetry. *J. Amer. Statist. Assoc.* 77, 639–646.
 Bickel, P.J., Lehmann, E.L., 1975. Descriptive statistics for nonparametric models. *Ann. Statist.* 3, 1038–1069.
 Cabilio, P., Masaro, J., 1996. A simple test of symmetry about an unknown median. *Canad. J. Statist.* 24, 349–361.
 Cantelli, F.P., 1933. Sulla determinazione empirica delle leggi di probabilita. *Giorn. Ist. Ital. Attuari* 4, 421–424.
 Csörgő, S., Heathcote, C.R., 1987. Testing for symmetry. *Biometrika* 74, 177–184.
 Feuerverger, A., Mureika, R.A., 1977. The empirical characteristic function and its applications. *Ann. Statist.* 5, 88–97.
 Fraser, D.A.S., 1957. *Nonparametric Methods in Statistics*. Wiley, New York.
 Gastwirth, J.L., 1971. On the sign test for symmetry. *J. Amer. Statist. Assoc.* 66, 821–823.
 Glivenko, V., 1933. Sulla determinazione empirica della legge di probabilita. *Giorn. Ist. Ital. Attuari* 4, 92–99.

- Gupta, M.K., 1967. An asymptotically nonparametric test of symmetry. *Ann. Math. Statist.* 38, 849–866.
- MacGillivray, H.L., 1982. Skewness properties of asymmetric forms of Tukey lambda distributions. *Comm. Statist. Theory Methods* 11, 2239–2248.
- Mira, A., 1999. Distribution-free test for symmetry based on Bonferroni's measure. *J. Appl. Statist.* 26, 959–971.
- Randles, R., Flinger, M., Policello, G., Wolfe, D., 1980. An asymptotically distribution-free test for symmetry versus asymmetry. *J. Amer. Statist. Assoc.* 75, 168–172.
- van Es, B., 1992. Estimating functionals related to a density by a class of statistics based on spacings. *Scand. J. Statist.* 19, 61–72.
- Vasicek, O., 1976. A test for normality based on sample entropy. *J. Roy. Statist. Soc. Ser. B* 38, 54–59.
- Yu, J., 1986. Consistency of an alternative nearest neighbour probability density function estimator. *Acta Math. Sci.* 6, 467–477.
- Yu, J., 1994. Consistency of a nearest neighbor density estimator for dependent variables. Research Report No. 1, Department of Mathematical Statistics, Umeå University.
- Yu, J., Ekström, M., 2000. Asymptotic properties of high order spacings under dependence assumptions. *Math. Methods Statist.* 9, 437–448.